P. S. Since the writing of the above, I have been inform'd, that, in two or three houses, singing birds, which were at the time roosting in their cages, were thrown off their perches by the suddenness of the shock.

LXXVI. *An Investigation of some Theorems which suggest some remarkable Properties of the Circle, and are of Use in resolving Fractions, whose Denominators are certain Multinomials, into more simple ones. By Mr.* John Landen.

Read May 2, 1754.

THAT the principal theorems, below investigated, will be of considerable use in the doctrine of fluxions, by rendering, in many cases, the business of computing fluents more easy, will, on perusal, be obvious to every one acquainted with that branch of science. Therefore, as the facilitating computations in that doctrine (which affords us wonderful assistance in many physical enquiries) may be a means of extending our knowlege in philosophy; it is presum'd, that this paper will not be thought unworthy the notice of the Royal Society.

I.

Supposing $\frac{n\dot{x}}{\sqrt{x^2-1}} = \frac{\dot{y}}{\sqrt{y^2-1}}$, where $\dot{x}$ and $\dot{y}$ denote the fluxions of the variable quantities x and y re-

respectively, and n an invariable quantity; it is propos'd to find, in terms of y and z, the equation of which z is a root, and $z^2 - 2xz + 1 = 0$, a divisor.

Taking the fluents of the given fluxionary equation, we have, supposing $x = 1$ when y is $= 1$, hyp. log. of $\overline{x + \sqrt{x^2 - 1}}^{\,n} =$ hyp. log. of $y + \sqrt{y^2 - 1}$, or $\overline{x + \sqrt{x^2 - 1}}^{\,n} = y + \sqrt{y^2 - 1}$: Whence, by substituting for x its value $\frac{z^2 + 1}{2z}$ (found by the equation $z^2 - 2xz + 1 = 0$), we have $z^n = y + \sqrt{y^2 - 1}$: Therefore $z^n - y$ is $= \sqrt{y^2 - 1}$; and, squaring both sides, $z^{2n} - 2yz^n + y^2 = y^2 - 1$. Consequently $z^{2n} - 2yz^n + 1$ is $= 0$; which, supposing n a positive integer, is the equation sought.

Now it is obvious, n being such an integer, that this equation will have as many trinomial divisors, of the form $z^2 - 2xz + 1$, as there are values of x corresponding to a given value of y: Which values of x, when y is not greater than 1, nor less than -1 (the only case I purpose to consider), will not be readily obtain'd from the equation $\overline{x + \sqrt{x^2 - 1}}^{\,n} = y + \sqrt{y^2 - 1}$ found above: But, if we multiply the given fluxionary equation by $\frac{1}{\sqrt{-1}}$, we get $\frac{n\dot{x}}{\sqrt{1 - x^2}} = \frac{\dot{y}}{\sqrt{1 - y^2}}$; of which the equation of the fluents is $n \times$ circ. arc rad. 1. cosine $x =$ circ. arc rad. 1. cosine y; where x is $= 1$ when y is $= 1$, agreeable to the supposition we made

made above when we took the fluents of the given fluxionary equation by logarithms. Therefore if A be put for the leaſt arc whoſe coſine is y, and C for the whole circumference, radius being 1; y being the coſine of A, $A+C$, $A+2C$, $A+3C$, &c. x will be the coſine of $\frac{A}{n}$, $\frac{A+C}{n}$, $\frac{A+2C}{n}$, &c. . . . . to $\frac{A+\overline{n-1}\times C}{n}$.

Conſequently, expreſſing the laſt-mention'd coſines, or the ſeveral values of x, by p, q, r, s, &c. $z^{2n}-2yz^{n}+1$ will be $=\overline{z^2-2pz+1}\times\overline{z^2-2qz+1}\times\overline{z^2-2rz+1}$, &c. ($n$), when n is a poſitive integer (as we ſhall always ſuppoſe it to be), let z be what it will.

Hence may be eaſily deduc'd a demonſtration of that remarkable property of the circle firſt diſcover'd by Mr. Cotes: But as that property has already been demonſtrated by ſeveral mathematicians, I ſhall omit taking any farther notice of it, and proceed in the inveſtigation of ſome other uſeful theorems which I do not find have ever yet been publiſh'd.

II.

If y be $=1$; then, A being $=0$; p, q, r, &c. will be the coſines of $\frac{0}{n}$, $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c. (n) reſpectively: Therefore p will be $=1$; and, if n be an even number, one of the coſines q, r, s, &c. will be $=-1$, one of the arcs $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c. being then $=\frac{C}{2}$.

III.

III.

If y be $= -1$; then, A being $= \frac{C}{2}$; p, q, r, s, &c. will be the cosines of $\frac{C}{2n}$, $\frac{3C}{2n}$, $\frac{5C}{2n}$, &c. (n) respectively: Therefore, if n be an odd number, one of those arcs will be $\frac{C}{2}$, whose cosine is -1.

IV.

If in the equations $z^{2n} - 2yz^n + 1 = 0$, and $z^2 - 2xz + 1 = 0$, we substitute $v - 1$ for z, they become $\overline{v-1}^{2n} - 2y \times \overline{v-1}^{n} + 1 = 0$, and $\overline{v-1}^{2} - 2x \times \overline{v-1} + 1 = v^2 - \overline{2+2x} \times v + 2 + 2x = 0$. Consequently

$$\left.\begin{array}{r} v^{2n} - 2nv^{2n-1} + \ldots + 2n \times \frac{2n-1}{2} v^2 - 2nv + 1 \\ \ldots \mp 2yn \times \frac{n-1}{2} v^2 \pm 2ynv \mp 2y \\ +1 \end{array}\right\} =$$

$$\overline{v^2 - \overline{2+2p} \times v + 2 + 2p} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q} \times \overline{v^2 - \overline{2+2r} \times v + 2 + 2r} \times \text{\&c. } (n);$$

where, of the two signs prefix'd to the terms where y is a factor, the upper or lower takes place, according as n is an even or an odd number. Whence, by the nature of equations, it follows, that $\overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. is $= 2 \mp 2y$. But this equation vanishing when y is $= 1$ and n an even number, or when y is $= -1$

 and

and n an odd number, it will be proper to consider those two cases more particularly.

V.

First, Let us suppose $y = 1$, and n an even number: Then p being $= 1$, and one of the other cosines q, r, s, &c. $= -1$ (*Art.* II.); we shall have

$$v^{2n} - 2nv^{2n-1} + \ldots + n^2v^2 = \overline{v^2+0} \times \overline{v^2 - 4v + 4} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q} \times \overline{v^2 - \overline{2+2r} \times v + 2 + 2r}, \text{ \&c.}$$

Therefore dividing by v^2,

$$v^{2n-2} - 2nv^{2n-3} + \ldots + n^2 = \overline{v^2 - 4v + 4} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q} \times \overline{v^2 - \overline{2+2r} \times v + 2 + 2r},$$

&c. that factor in which the value of the cosine q, or r, &c. is -1, being expung'd.

Consequently n^2 is $= 4 \times \overline{2+2q} \times \overline{2+2r} \times \overline{2+2s}$, &c. when the factor, whose value is nothing, is expung'd.

VI.

Let us now suppose $y = -1$, and n an odd number: Then one of the cosines p, q, r, &c. being $= -1$ (*Art.* III.),

$v^{2n} - 2nv^{2n-1} + \ldots + n^2v^2$ will be $= \overline{v^2+0} \times \overline{v^2 - \overline{2+2p} \times v + 2 + 2p} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q}$, &c. Therefore, dividing by v^2,

$v^{2n-2} - 2nv^{2n-3} + \ldots + n^2$ will be $= \overline{v^2 - \overline{2+2p} \times v + 2 + 2p} \times \overline{v^2 - \overline{2+2q} \times v + 2 + 2q}$, &c. and consequently $n^2 = \overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. when the factor, whose value is nothing, is expung'd.

VII.

VII.

Subſtituting in the equations $z^{2n} - 2yz^{n} + 1 = 0$, and $z^{2} - 2xz + 1 = 0$, $\frac{a+\omega}{a-\omega}$ inſtead of z, we have

$$\overline{\frac{a+\omega}{a-\omega}}\Big|^{2n} - 2y \times \overline{\frac{a+\omega}{a-\omega}}\Big|^{n} + 1$$

$$= \frac{\overline{a+\omega}^{2n} - 2y \times \overline{a+\omega}^{n} \times \overline{a-\omega}^{n} + \overline{a-\omega}^{2n}}{\overline{a-\omega}^{2n}} = 0, \text{ and}$$

$$\overline{\frac{a+\omega}{a-\omega}}\Big|^{2} - 2x \times \frac{a+\omega}{a-\omega} + 1$$

$$= \frac{\overline{a+\omega}^{2} - 2x \times \overline{a+\omega} \times \overline{a-\omega} + \overline{a-\omega}^{2}}{\overline{a-\omega}^{2}}$$

$$= \frac{\overline{2+2x} \times \overline{\omega^{2} + \frac{1-x}{1+x} a^{2}}}{\overline{a-\omega}^{2}} = 0. \quad \text{Conſequently}$$

$\overline{a+\omega}^{2n} - 2y \times \overline{a+\omega}^{n} \times \overline{a-\omega}^{n} + \overline{a-\omega}^{2n}$ will be $=$ $\overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. $\times\, \overline{\omega^{2} + \frac{1-p}{1+p} a^{2}}$ $\times\, \overline{\omega^{2} + \frac{1-q}{1+q} a^{2}} \times \overline{\omega^{2} + \frac{1-r}{1+r} a^{2}}$, &c.

But, by *Art.* IV. $\overline{2+2p} \times \overline{2+2q} \times \overline{2+2r}$, &c. is $= 2 \mp 2y$, the upper or lower of the two ſigns prefix'd

prefix'd to y taking place according as n is an even or an odd number.

Therefore $\overline{a+\omega}^{2n} - 2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{n}$ is $=$ $\overline{2+2y}\times\overline{\omega^2+\frac{1-p}{1+p}a^2}\times\overline{\omega^2+\frac{1-q}{1+q}a^2}\times\overline{\omega^2+\frac{1-r}{1+r}a^2}$, &c.

Now p being the cosine of any number of degrees, radius being 1, $\frac{1-p}{1+p}a^2$ will be the square of the tangent of half so many degrees, radius being a: Therefore, denoting that tangent by b; and the tangents of half the arcs describ'd with the radius a, whose cosines, when the radius is 1, are q, r, s, &c. being denoted by c, d, e, &c. respectively; we have $\overline{a+\omega}^{2n} - 2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n} = \overline{2\mp 2y}\times\overline{\omega^2+b^2}\times\overline{\omega^2+c^2}\times\overline{\omega^2+d^2}$, &c. But when y is $=1$, and n an even number; or $y=-1$, and n an odd number; $2\mp 2y$ being $=0$; nothing can be determin'd from that equation: Wherefore, in those cases, recourse must be had to what is done above.

VIII.

Let us suppose $y=1$, and n an even number: Then the equation $\overline{a+\omega}^{2n} - 2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n} = \overline{2+2p}\times\overline{2+2q}\times\overline{2+2r}$, &c. $\times\overline{\omega^2+\frac{1-p}{1+p}a^2}\times\overline{\omega^2+\frac{1-q}{1+q}a^2}\times\overline{\omega^2+\frac{1-r}{1+r}a^2}$, &c.

becomes

becomes $\overline{a+\omega}^{2n} - 2\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n}=4$ $\times\overline{2+2q}\times\overline{2+2r}$, &c. $\omega^2\times\overline{\omega^2+\frac{1-q}{1+q}a^2}\times\overline{\omega^2+\frac{1-r}{1+r}a^2}$, &c. p being $=1$ (*Art.* II.) and $\frac{1-p}{1+p}a^2$ $(=b^2)=0$.

Moreover, one of the other cosines q, r, s, &c. being $=-1$ (*Art.* II.), some one of the factors $2+2q$, $2+2r$, $2+2s$, &c. will vanish: Which factor being expung'd from the product $4\times\overline{2+2q}\times\overline{2+2r}$, &c. and restor'd to the divisor $\omega^2+\frac{1-q}{1+q}a^2$, or $\omega^2+\frac{1-r}{1+r}a^2$, &c. from which it was taken, that divisor will become $4a^2$; and the product $4\times\overline{2+2q}\times\overline{2+2r}$, &c. will then (by *Art.* V.) be $=n^2$.

Consequently $\overline{a+\omega}^{2n}-2\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n}$, will be $=n^2\times\omega^2\times 4a^2\times\overline{\omega^2+c^2}\times\overline{\omega^2+d^2}$, &c. where the factor $4a^2$ takes place instead of ω^2+ sq. of the tang. of 90°.

If y be $=1$, and n an odd number, p will be $=1$, and $b=0$; but no one of the cosines q, r, s, &c. will be $=-1$, as when n is an even number. Therefore, in this case, the equation $\overline{a+\omega}^{2n}-2y\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{2n}=\overline{2+2y}\times\overline{\omega^2+b^2}\times\overline{\omega^2+c^2}$, &c. becomes $\overline{a+\omega}^{2n}-2\times\overline{a+\omega}^{n}\times\overline{a-\omega}^{n}+\overline{a-\omega}^{n}=$ $4\times\omega^2\times\overline{\omega^2+c^2}\times\overline{\omega^2+d^2}$, &c.

IX.

IX.

By taking the ſquare root of $\overline{a+\omega}^{2n} - 2\times\overline{a+\omega}^{n} \times\overline{a-\omega}^{n} + \overline{a-\omega}^{2n}$, and of its two values juſt now found, we have, when n is an even number, $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2\,a\,n\,\omega\times\sqrt{\omega^2+c^2}\times\sqrt{\omega^2+d^2}$, &c. $2\,a$ taking place inſtead of $\sqrt{\omega^2 + \text{ſq. of the tang. of } 90°}$.

And, when n is an odd number, $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2\,\omega\times\sqrt{\omega^2+c^2}\times\sqrt{\omega^2+d^2}$, &c. Whence the following conſtruction is inferr'd.

X.

Deſcribe about the centre C (*Plate* XX. *fig.* 1. and 2.), with the radius a, the circle $P\,A'\,A''\,A'''$, &c.; draw the diameter $P\,C\,Q$, and the tangent $B'''\,P\,B^5$; divide the ſemicircumference $P\,A'\,Q$ into as many equal parts $P\,A'$, $A'\,A''$, $A''\,A'''$, &c. as there are units in the integer n; draw the ſecants $C\,A'\,B'$, $C\,A''\,B''$, &c. and, taking on $C\,Q$ any point O, draw $K'''\,O\,K^5$ parallel to $B'''\,P\,B^5$; likewiſe draw $B'\,K'$, $B''\,K''$, $B'''\,K'''$, &c. parallel to $P\,Q$; and call $C\,O$, ω.

Then will q be the coſine of twice the angle $P\,C\,A'$, r the coſine of twice $P\,C\,A''$, s the coſine of twice $P\,C\,A'''$, &c. if the radius be 1.

Therefore $P\,B' = O\,K'$ will be $= c$, $P\,B'' = O\,K'' = d$, &c. and $C\,K' = \sqrt{\omega^2+c^2}$, $C\,K'' = \sqrt{\omega^2+d^2}$, &c. Conſequently $O\,P^n - O\,Q^n$ being $= \overline{a+\omega}^{n}$

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$-\overline{a-\omega}^{n}$, and $n \times PQ \times CO \times CK' \times CK''$, &c. $=$ $2an\omega \times \sqrt{\omega^2 + c^2} \times \sqrt{\omega^2 + d^2}$, &c. when n is an even number; $OP^n - OQ^n$ will then be $= n \times PQ \times CO \times CK' \times CK''$, &c. where the diameter PQ takes place inftead of the infinite quantity $CK^{\frac{n}{2}}$.

But if n be an odd number, $OP^n - OQ^n$ will be $= 2 \times CO \times CK' \times CK'' \times CK'''$, &c.

XI.

It is evident that, of the factors CK', CK'', CK''', &c. the firft and laft, the fecond and laft but one, &c. are refpectively equal to each other. Therefore, omitting the fquares of the factors below PQ, and the fquares of their values,

$OP^n - OQ^n$ is $= n \times PQ \times CO \times CK'^2 \times CK''^2 \times CK'''^2$, &c. and $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2an\omega \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$, &c. when n is an even number; or $OP^n - OQ^n$ is $= 2 \times CO \times CK'^2 \times CK''^2 \times CK'''^2$, &c. and $\overline{a+\omega}^{n} - \overline{a-\omega}^{n} = 2\omega \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$, &c. when n is an odd number.

XII.

If we fuppofe $y = -1$, and n an odd number, it will appear, by proceeding much in the fame manner as in *Art.* VIII. that $\overline{a+\omega}^{2n} + 2 \times \overline{a+\omega}^{n} \times \overline{a-\omega}^{n} + \overline{a-\omega}^{2n}$ is $= n^2 \times 4a^2 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$, &c.

&c. where the factor $4a^2$ takes place instead of $\omega^2 +$ sq. of the tang. of $90°$.

If y be $= - 1$, and n an even number, $\overline{a+\omega}^{2n} + 2 \times \overline{a+\omega}^{n} \times \overline{a-\omega}^{n} + \overline{a-\omega}^{2n}$ is $= 4 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$, &c.

Whence, by extracting the square root of both sides of those equations, we have, when n is an odd number, $\overline{a+\omega}^{n} + \overline{a-\omega}^{n} = 2\,a\,n \times \sqrt{\omega^2 + b^2} \times \sqrt{\omega^2 + c^2}$, &c. $2\,a$ taking place instead of $\sqrt{\omega^2 + \text{sq. of the tang. of } 90°}$: And, when n is an even number, $\overline{a+\omega}^{n} + \overline{a-\omega}^{n} = 2 \times \sqrt{\omega^2 + b^2} \times \sqrt{\omega^2 + c^2}$, &c. Hence we infer this construction.

XIII.

Having describ'd about the centre C (*fig.* 3. and 4.), with the radius a, the circle $P\,a'\,A'\,a''\,A''$, &c. draw the diameter $P\,C\,Q$, and the tangent $b''\,P\,b^4$; divide the semicircumference $P\,a'\,Q$ into as many equal parts $P\,a'$, $a'\,A'$, $A'\,a''$, &c. as there are units in $2\,n$; draw the secants $C\,a'\,b'$, $C\,a''\,b''$, &c. and, thro' any point (O) in $C\,Q$, draw $k''\,O\,k^4$ parallel to $b''\,P\,b^4$; likewise draw $b'\,k'$, $b''\,k''$, &c. parallel to $P\,Q$; and call $C\,O$, ω.

Then, if the radius be 1, p will be the cosine of twice the angle $P\,C\,a'$, q the cosine of twice $P\,C\,a''$, &c. therefore $P\,b' = O\,k'$ will be $= b$, $P\,b'' = O\,k''$ $= c$, &c. and $C\,k' = \sqrt{\omega^2 + b^2}$, $C\,k'' = \sqrt{\omega^2 + c^2}$, &c.

Con-

Consequently $O\,P^n + O\,Q^n$ being $= \overline{a+\omega}^n + \overline{a-\omega}^n$, and $n \times P\,Q \times C\,k' \times C\,k''$, &c. $= 2an \times \sqrt{\omega^2 + b^2} \times \sqrt{\omega^2 + c^2}$, &c. when n is an odd number; $O\,P^n + O\,Q^n$ will then be $= n \times P\,Q \times C\,k' \times C\,k''$, &c. where the diameter $P\,Q$ takes place instead of the infinite quantity $C\,k^{\frac{n+1}{2}}$.

But if n be an even number, $O\,P^n + O\,Q^n$ will be $= 2 \times C\,k' \times C\,k''$, &c.

XIV.

It is obvious that, of the factors $C\,k'$, $C\,k''$, &c. the first and last, the second and last but one, &c. are respectively equal to each other: Therefore the squares of the factors below $P\,Q$, and the squares of their values, being omitted,

$O\,P^n + O\,Q^n$ is $= n \times P\,Q \times C\,k'^2 \times C\,k''^2$, &c. and

$\overline{a+\omega}^n + \overline{a-\omega}^n = 2\,a\,n \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$, &c. when n is an odd number; or

$O\,P^n + O\,Q^n$ is $= 2 \times C\,k'^2 \times C\,k''^2$, &c. and

$\overline{a+\omega}^n + \overline{a-\omega}^n = 2 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$, &c. when n is an even number.

XV.

Writing in the equation $\overline{a+\omega}^{2n} - 2\,y \times \overline{a+\omega} \times \overline{a-\omega}^n + \overline{a-\omega}^{2n} = \overline{2 \mp 2\,y} \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$, &c.

 found

(found by *Art.* VII.) $a-u$ for ω, the same becomes $\overline{2a-u}^{2n} - 2yu^{n} \times \overline{2a-u}^{n} + u^{2n} = \overline{2 \mp 2y} \times \overline{u^2 - 2au + a^2 + b^2} \times \overline{u^2 - 2au + a^2 + c^2}$, &c. $= \overline{2 \mp 2y} \times \overline{u^2 - 2au + \beta^2} \times \overline{u^2 - 2au + \gamma^2} \times \overline{u^2 - 2au + \delta^2}$, &c. if instead of $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$, &c. (the secants of the arcs of which b, c, d, &c. are tangents), we put β, γ, δ, &c.

And, by a like substitution in the equations in *Art.* XI. and XIV. it appears, that $\overline{2a-u}^{n} - u^{n}$ is $= 2an \times \overline{a-u} \times \overline{u^2 - 2au + \gamma^2} \times \overline{u^2 - 2au + \delta^2}$, &c. or $2 \times \overline{a-u} \times \overline{u^2 - 2au + \gamma^2} \times \overline{u^2 - 2au + \delta^2}$, &c. according as n is an even or an odd number: And that $\overline{2a-u}^{n} + u^{n}$ is $= 2an \times \overline{u^2 - 2au + \beta^2} \times \overline{n^2 - 2au + \gamma^2}$, &c. or $2 \times \overline{u^2 - 2au + \beta^2} \times \overline{u^2 - 2au + \gamma^2}$, &c. according as n is an odd or an even number.

From what is done above, I might now deduce many corollaries; and, by means of other substitutions, investigate other theorems; but want of leisure obliges me to desist.

LXXVII. *An*

Fig. 1.

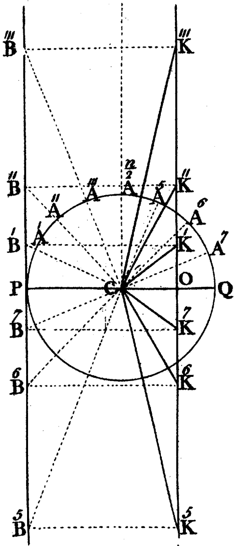


Fig. 2.

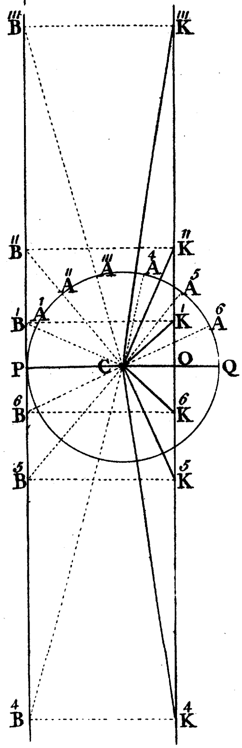


Fig. 3.

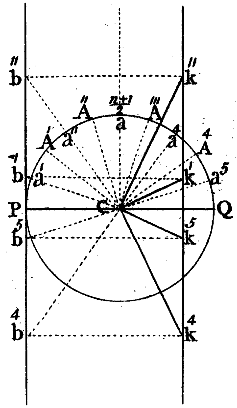
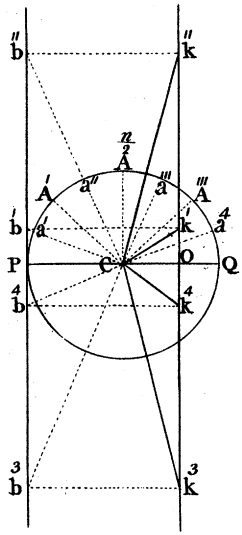


Fig. 4.



J. Mynde sc.